ELSEVIER

Available online at www.sciencedirect.com

SciVerse ScienceDirect

Procedia - Social and Behavioral Sciences 54 (2012) 907 – 916

Procedia
Social and Behavioral Sciences

EWGT 2012

15th meeting of the EURO Working Group on Transportation

Numerical solutions to the Logit lane assignment model

M.M. Khoshyaran[a], J.P. Lebacque[b,*]

[a]*ETC Economics Traffic Clinic, 34 av des Champs Elysées, 75008 PARIS , France, etclinic@wanadoo.fr,*
[b]*UPE IFSTTAR GRETTIA, Le Descartes 2, rue de la Butte Verte, 77166 MARNE-LA-VALLEE, France*

Abstract

The Logit lane assignment model is a macroscopic model for multi-lane flow. This model assumes that drivers evaluate the utility of available lanes, and that each driver chooses the lane with highest utility. The utility of a lane increases with the speed of traffic, and is stochastic with Gumbel random component. Since the speed of traffic on a lane, and hence the attractively of a lane, decreases with the number of cars using the lane, a (dynamic) equilibrium forms. This equilibrium results in the lane choice of drivers and the assignment of traffic between lanes. Formally it is a Logit model. The traffic flow is assumed multi-class, and FIFO on each lane, but not all lanes are available to all driver classes. The paper will address the homogeneous case, far from on- and off-ramps or intersections, where all lanes are accessible to all users. It will be shown that in the homogeneous case the total density follows a LWR model even though all lane speeds are different. The model can be expressed as a system of conservation laws with a smooth flux function, but analytical solutions are generally not available with very few exceptions. Thus numerical methods are required in order to use the model for applications. The paper will explore numerical methods in the case of a regular section.

Keywords: macroscopic traffic modeling, traffic, multilane, stochastic, logit model, lane assignment, lagrangian scheme, particl discretization

1. Introduction

The need for multilane traffic flow modeling is important.

- Simulation, planning and assessment of traffic on infrastructures, such as peri-urban motorways which feature a large number of merges and diverges. These merges and diverges induce lane-specific traffic dynamics.

* Corresponding author. Tel.: +33 1 45925626.
E-mail address: jean-patrick.lebacque@ifsttar.fr

 doi:10.1016/j.sbspro.2012.09.806

- Urban streets and arterial networks are endowed with many complex intersections. These also induce lane-specific traffic dynamics.
- Managed lanes require lane-specific control and traffic management measures and thus require multi-lane models for the implementation and assessment of their operations.

Multi-lane traffic exhibits dynamical properties which can be significantly different on each lane. Indeed it is a standard observation that even the speed-density or the flow-density fundamental diagrams differ from a lane to another. The figure 1 illustrates this fact.

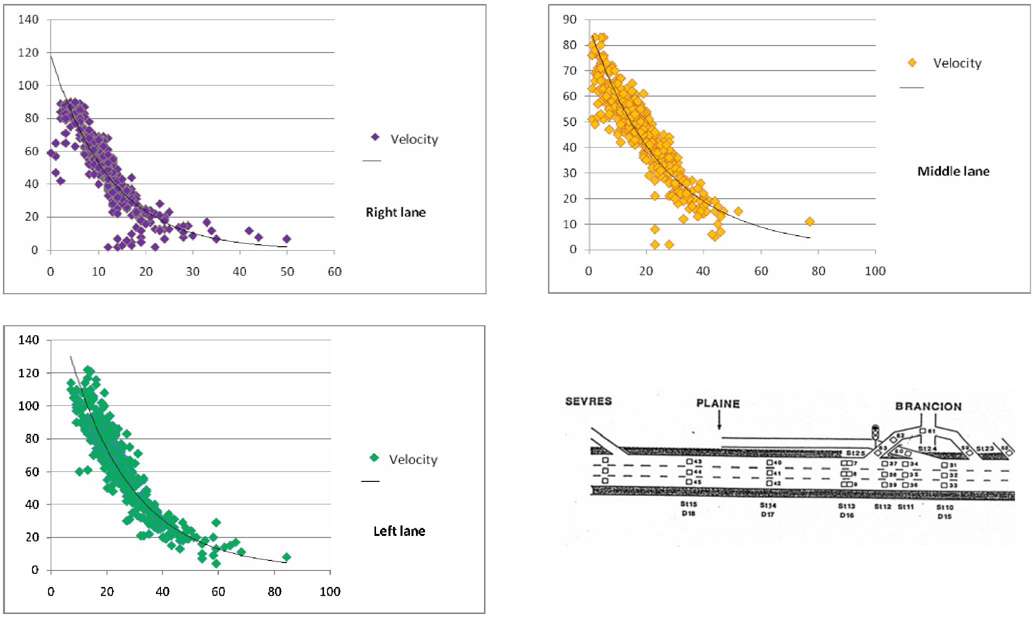


Figure 1: lane-dependent speed-density fundamental diagrams (Bd Périphérique, Paris, 20 seconds -detector data for detector D17 half-way between Brancion and Sèvres). Horizontal axis: occupancy (in %), vertical axis: velocity (km/hr)

Traffic dynamics in a multi-lane context depend on many factors: lane-change behavior, lane-specific fundamental diagrams, driver destination, and lane management. Many models have been proposed in order to address multi-lane modeling. The model proposed in this paper is macroscopic, based on lane densities and densities disaggregated per destination. The choice of a macroscopic model results from the observation that it is very difficult to collect the necessary data to feed a microscopic lane change and equilibrium model. Further, for traffic control and evaluation purposes a macroscopic model will be sufficient and easier to identify.

We assume that an equilibrium forms between lanes, resulting from the attractivity of lanes versus maneuver requirements induced by user purposes (destinations). The main determinant of lane choice is assumed to be the lane speed.

2. Some background and short literature review on multi-lane modeling.

There is a vast literature on micro multi-lane models. But we will only consider macroscopic multi-lane models, inasmuch as they are related to the model developed in this paper.

The most "macroscopic" of multi-lane models do not attempt to model the actual flows between lanes or the assignment between lanes. They simply describe the macroscopic impact of lane dynamics on the global flow (Greenberg et al 2003, Jin 2006).

At the other end of the scale, many models have been developed, which aim to describe the actual interaction between traffic on lanes. Let us mention Michalopoulos et al. 1984 based on a relaxation process tending to equilibrate lane densities. An approach describing flows between lanes (based on differences in speed) and the impacts of lane change, has been developed in a series of papers, refer for instance to Laval and Daganzo 2005 and 2006, Laval and Leclercq 2008, see also Schnetzler et al 2010. Kinetic models propose approaches which integrate microscopic lane change behavior into macroscopic models, beginning with Helbing 1997, see also Hoogendoorn and Bovy 1999, Ngoduy 2006 with careful inclusion of lane change motives.

At an intermediate level a relatively precise description can be achieved by considering the net result of flows between lanes without describing the details of the dynamics which lead to this net result. If we consider only traffic state on lanes we can assume that there is an equilibrium between lanes. Daganzo 1997 initiated this approach in the case of special lanes and two classes of vehicles based on an all or nothing assignment. Lebacque and Khoshyaran 1998-2002 proposed an extension to many lanes and vehicle classes. This model is expressed as a system of conservation laws, the flow function of which is defined implicitly, and admits only a piece-wise continuous gradient. The model presented in this paper expands on paper Lebacque and Khoshyaran 2009. It assumes a stochastic utility as basis for lane choice and yields a model more general, more regular and tractable, and better supported by data, than the deterministic lane equilibrium model.

3. The model

3.1. Notations and definitions

The set of lanes is I, the set of user classes is D, lanes are denoted $i \in I$. The set of lanes accessible to users d is denoted I^d. The set of user classes that can access lane i is denoted D_i .These sets depend on the location x. Densities used throughout the paper are

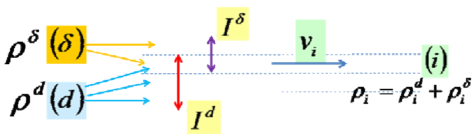
- $\rho^d(x,t)$: density of vehicles of class d at time t and location x
- $\rho_i^d(x,t)$: density of vehicles of class d using lane i at time t and location x
- $\rho_i(x,t)$: density of vehicles on lane i at time t and location x

The conserved variables are the partial densities $\rho^d(x,t)$. The unknowns are the densities $\rho_i^d(x,t)$ which describe how the densities per class are assigned between lanes. Thus:

$$\rho^d = \sum_{i \in I^d} \rho_i^d$$

The lane densities result according to

$$\rho_i \stackrel{def}{=} \sum_{d \in D_i} \rho_i^d$$



Finally we assume a lane-specific fundamental diagram; the velocity of vehicles on lane i is given by

$$v_i = V_i(\rho_i)$$

3.2. Logit lane assignment

In order to estimate how drivers assign themselves between lanes, we make the following assumptions:

- The actual dynamics of lane change are neglected; only the net result is considered. This hypothesis concerns the resolution of the model which cannot be high. The model is purely macroscopic.
- Drivers chose the lane they consider the best. They evaluate the quality of a lane following a stochastic utility function.
- The utility of lane $i \in I^d$ for a driver of class $d \in D$ is given by the following:

$$U_i^d \stackrel{def}{=} v_i + \mu_i^d + \xi_i^d \qquad (1)$$

with $v_i = V_i(\rho_i)$ the speed of traffic on lane i, μ_i^d a constant expressing a preference of drivers d for lane i (for instance a right lane for drivers expecting to exit the highway soon). Finally ξ_i^d denotes a Gumbel random variable with standard deviation ϑ expressing all stochastic elements of the lane choice: variability of drivers in terms of perception and reaction, random perturbations of traffic, impact of "hidden variables" i.e. all processes which are not included in the model.

The Logit model for lane assignment results:

$$\begin{aligned}
&\rho^d = \sum_{i\in I^d} \rho_i^d \qquad \forall d \in D \\
&v_i = V_i(\rho_i) \quad with\ \rho_i \overset{def}{=} \sum_{d\in ID_i} \rho_i^d \quad \forall i \in I \\
&\frac{\rho_i^d}{\rho^d} = \frac{\exp\left(\dfrac{v_i + \mu_i^d}{\vartheta}\right)}{\sum_{\ell\in I^d} \exp\left(\dfrac{v_\ell + \mu_\ell^d}{\vartheta}\right)} \quad \forall d \in D, \forall i \in I^d
\end{aligned} \tag{2}$$

This model can be interpreted as a fixed point problem in which the partial densities $\rho^d, d \in D$ constitute the data, and the partial densities $\rho_i^d, d \in D, i \in I^d$ the unknowns. The lane densities are derived variables.

3.3. The general model

Let us introduce $H(x)$ the neguentropy function, depicted on figure 2.

$$H(x) \overset{def}{=} x(\ln(x) - 1)$$

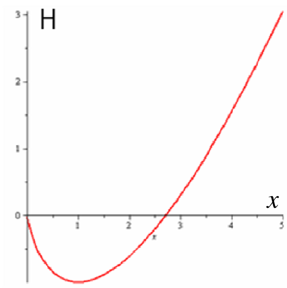


Figure 2: the neguentropy function

It can be shown the solution of the fixed point problem (2) is identical with the solution of the following linear-concave optimization problem

$$\begin{aligned}
&\max \sum_{i\in I^d} \int_0^{\rho_i} V_i(r)dr + \sum_{d\in D, i\in I^d} \rho_i^d \mu_i^d - \vartheta \sum_{d\in D, i\in I^d} \rho^d H\left(\frac{\rho_i^d}{\rho^d}\right) \\
&\left|\ \rho^d = \sum_{i\in I^d} \rho_i^d \qquad \forall d \in D\right.
\end{aligned} \tag{3}$$

It suffices to express the Karush-Kuhn-Tucker optimality conditions of (3); they can be recast as (2). From the unicity of the solution of (2) and the invertibility of the hessian of the criterion in (3) we deduce that the partial densities $\rho_i^d, d \in D, i \in I_d$ are smooth functions of the class densities $\rho^d, d \in D$. Let us define:

$$r \overset{def}{=} \left(\rho^d\right)_{d\in D} \quad \text{and} \quad R \overset{def}{=} \left(\rho_i^d\right)_{d\in D, i\in I^d}$$

then:

$$R = \Phi(r) \quad \text{i.e.} \quad \rho_i^d = \Phi_i^d(r) \quad \forall d \in D, \forall i \in I^d$$

where Φ_i^d are smooth functions determined by (2). The class densities $\rho^d, d \in D$ satisfy the following system of conservation equations:

$$\partial_t r + \partial_x f(r) = 0 \tag{4}$$

with $f(r)^d \overset{def}{=} \sum_{i \in I^d} \rho_i^d V_i(\rho_i) = \sum_{i \in I^d} \Phi_i^d(r) V_i\left(\sum_{\delta \in D_i} \Phi_i^\delta(r)\right)$. f is smooth but defined implicitly.

4. The homogeneous case

4.1. Definition

At locations far from in- and on-ramps or intersections, all lanes are available to all user classes. Further, the preference for a given lane will not depend on the driver class. Such a situation will be called "homogeneous" with respect to lane-assignment. Thus the homogeneous case is characterized by the following:

$$\left| \begin{array}{l} D_i = D \quad \forall i \in I \quad i.e. \quad I^d = I \quad \forall d \in D \\ \mu_i^d = \mu_i \quad \forall i \in I, \quad \forall d \in D \end{array} \right. \tag{5}$$

It follows that the assignment of traffic between lanes does not depend on the driver class d:

$$\frac{\rho_i^d}{\rho^d} = \frac{\rho_i}{\rho} = \frac{\exp\left(\frac{v_i + \mu_i}{\vartheta}\right)}{\sum_{\ell \in I} \exp\left(\frac{v_\ell + \mu_\ell}{\vartheta}\right)} \quad \forall d \in D, \forall i \in I$$

with $\rho_i = \sum_{d \in D} \rho_i^d$ the density on lane i and $\rho = \sum_{i \in I} \rho_i = \sum_{d \in D} \rho^d = \sum_{d \in D, i \in I} \rho_i^d$ the total density. The speed of traffic on lane i is $v_i = V_i(\rho_i)$ Thus the lane densities (6) are solution of

$$\begin{array}{l} \max \sum_{i \in I} \int_0^{\rho_i} V_i(r) dr + \sum_{i \in I} \rho_i \mu_i - \vartheta \sum_{i \in I} \rho H\left(\frac{\rho_i}{\rho}\right) \\ \left| \rho = \sum_{i \in I} \rho_i \right. \end{array} \tag{7}$$

which is a linear-strictly concave program

4.2. Basic properties

It follows that the lane assignment (6), (7) can be expressed by a set of functions ψ_i such that:

$$\begin{array}{ll} \rho_i = \rho \psi_i(\rho) & \forall i \in I \\ \rho_i^d = \rho^d \psi_i(\rho) & \forall d \in D, \forall i \in I \end{array} \tag{8}$$

The functions ψ_i are smooth and defined implicitly. It follows that the total density satisfies a single conservation equation:

$$\partial_t \rho + \partial_x \rho V(\rho) = 0 \tag{9}$$

with:

$$V(\rho) \overset{def}{=} \sum_{i \in I} \rho_i V_i(\rho_i) = \sum_{i \in I} \psi_i(\rho) V_i(\psi_i(\rho)) \tag{10}$$

$V(\rho)$ is the fundamental diagram for the total density. It is defined as the weighted mean of the lane fundamental diagrams. Thus in the homogeneous case, the total density follows a LWR model (Lighthill and Whitham 1955, Richards 1956) although the speed of traffic and the fundamental diagram are different for each lane.

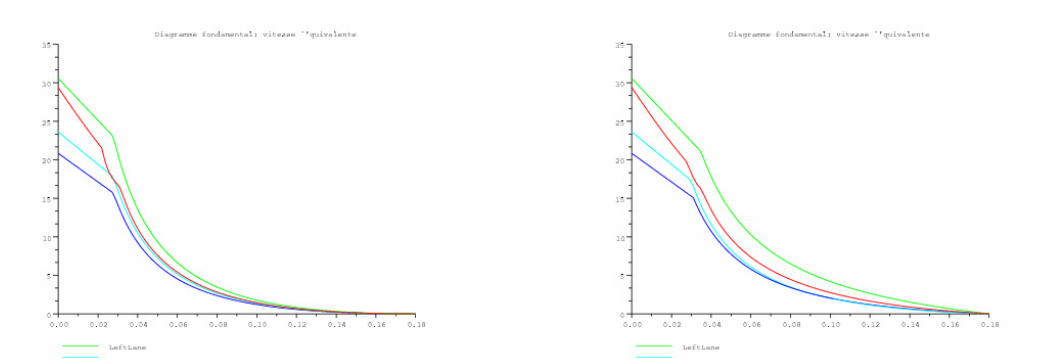


Figure 3: lane speed-density fundamental diagrams (blue: right, turquoise: middle, green: left lane). In red: the aggregated fundamental diagram for the total density. Traffic flow model: 1-phase Colombo model (Lebacque et al 2007). Horizontal axis: density in vh/m, vertical axis: velocity in m/s

It suffices to solve (9) by the standard Godunov scheme or a similar algorithm Lebacque 1993, Buisson et al 1995-1996. The partial densities are deduced from the total density according to (8).

The relationship (6) can be used in order to test the Logit hypothesis in the homogeneous case:

$$\vartheta\left(\ln\left(\frac{\rho_i}{\rho_{max,i}}\right)-\ln\left(\frac{\rho_j}{\rho_{max,j}}\right)\right)=v_i-v_j+\mu_i-\mu_j$$

The following example is taken from data collected on the Boulevard Périphérique in Paris (site depicted on figure 1, detectors D17, half-way between Brancion and Sèvres. In spite of unfavourable conditions

- Close-by on- and off-ramps, thus the multi-lane traffic is not homogeneous
- Occupancy instead of density,
- 20 seconds sample time (which is short for an equilibrium model)

consistent estimates of ϑ are obtained (of the order of 13 km/h in the case of Bd Périphérique).

Extensive analysis (Schnetzler 2011) has shown that for instance the homogeneous Logit model applies well on the motorway A7 close to Marseilles but does not apply well on the A1 close to Paris. The difference between the two cases lies very likely with the fact that the A1 motorway exhibits a high density of on- and off-ramps, thus the homogeneous Logit lane assignment model cannot apply.

For infrastructures such as A1 it is necessary to develop numerical schemes in order to implement the full lane assignment model (2), (3), (4).

5. Numerical aspects

Calculation of the characteristic elements of (4) is difficult. Even estimating the gradient ∇f requires extensive algebra. Therefore most classical eulerian numerical solvers are difficult to implement and we concentrate on a lagrangian solver. The idea is that if r^d denotes the spacing of vehicles $d \in D$, v^d their mean speed, the conservation equation for vehicles $d \in D$ can be expressed as

$$\partial_t r^d + \partial_n v^d = 0 \tag{11}$$

with n the index of vehicles.

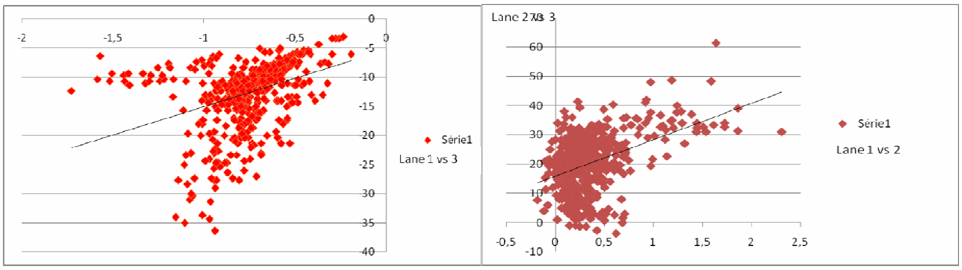


Figure 4: testing the Logit assignment hypothesis under the homogeneous flow assumption. The figure depicts the difference between speeds vs. the difference between logarithms of density. Two couples of lanes are considered left vs right and left vs middle. Horizontal axis: logarithm of density, vertical axis: velocities in km/hr

This formula simply means that the rate of variation of spacing is equal to the opposite of the gradient of velocity with respect to the index of vehicles.

In order to discretize (11), it is necessary to estimate the spacing r^d and the mean speed v^d.

5.1. Notations

Vehicles are discretized into groups (n,d) of vehicles of size ΔN_n^d with n the index of the vehicle group and $d \in D$ its class, and time is discretized time-steps of duration Δt. Densities and velocities are related by the following relationships.

$$\begin{aligned} \rho_i^d &= \rho^d \varphi_i^d \qquad \forall d \in D, \forall i \in I^d \\ \rho_i &= \sum_{d \in D_i} \rho^d \varphi_i^d \quad \forall i \in I \end{aligned} \tag{12}$$

Following (6), the coefficients φ_i^d are given by

$$\varphi_i^d = \frac{\exp\left(\dfrac{v_i + \mu_i}{\vartheta}\right)}{\sum_{\ell \in I^d} \exp\left(\dfrac{v_\ell + \mu_\ell}{\vartheta}\right)} \quad \forall d \in D, \forall i \in I \tag{13}$$

The coefficients φ_i^d allow us to estimate the velocities v^d:

$$v^d = \sum_{i \in I^d} \varphi_i^d V_i\left(\sum_{d \in I_i} \varphi_i^d \rho^d\right) \quad \forall d \in D \tag{14}$$

In order to estimate v^d in a lagrangian setting we use (14) and assume that each group (n,d) spreads between all available lanes $i \in I^d$. Let us introduce the following notations

- $x_n^d(t)$ the position of the rear of group (n,d) at time $t\,\Delta t$
- ΔN_n^d the size of group (n,d) (number of vehicles)
- $\varphi_{i,n}^d(t)$ the ratio of group (n,d) per lane $i \in I$
- On each lane $i \in I$, the group (n,d) admits a predecessor group $(m,e) \overset{def}{=} (M(n,d,i;t), E(n,d,i;t))$

Further we define

$$\Delta x_{i,n}^d(t) \overset{def}{=} x_m^e(t) - x_n^d(t) \qquad \text{with } (m,e) \overset{def}{=} (M(n,d,i;t), E(n,d,i;t)) \tag{15}$$

This quantity will serve to define the spacing as perceived at time $t\,\Delta t$ by the drivers in group (n,d) on the lane $i \in I$.

5.2. The algorithm

The first step consists in the calculation of the split coefficients $\varphi_{i,n}^{d}(t)$. Following (13) and (14), the split coefficients satisfy the following relationship:

$$\varphi_{i,n}^{d} = \frac{\exp\left(\frac{1}{\vartheta}\left[V_i\left(\Delta N_n^d\,\varphi_{i,n}^d \Big/ \Delta x_{i,n}^d(t)\right) + \mu_i\right]\right)}{\sum_{\ell \in I} \exp\left(\frac{1}{\vartheta}\left[V_\ell\left(\Delta N_n^d\,\varphi_{\ell,n}^d \Big/ \Delta x_{\ell,n}^d(t)\right) + \mu_\ell\right]\right)} \quad \forall d \in D, \forall i \in I \tag{16}$$

Equivalently the split coefficients $\varphi_{i,n}^{d}(t)$ are the unique solution of the following linear concave program:

$$\begin{aligned} &\max \sum_{i \in I^d} \int_0^{\varphi_{i,n}^d} V_i\left(\Delta N_{i,n}^d\,\varphi \Big/ \Delta x_{i,n}^d(t)\right) d\varphi + \sum_{i \in I^d} \mu_i\,\varphi_{i,n}^d - \vartheta \sum_{i \in I^d} H\left(\varphi_{i,n}^d\right) \\ &\Big|\; 1 = \sum_{i \in I^d} \varphi_{i,n}^d \end{aligned} \tag{17}$$

This second formulation is more tractable than the fixed point (16). It can be solved by a standard Newton algorithm.

As a **second step** we estimate the velocities at location $x_n^d(t)$

$$\begin{aligned} v_{i,n}^d(t) &= V_i\left(\Delta N_n^d\,\varphi_{i,n}^d(t) \Big/ \Delta x_{i,n}^d(t)\right) && \forall d \in D, \forall i \in I^d \\ v_n^d(t) &= \sum_{\forall i \in I^d} \varphi_{i,n}^d(t)\, v_{i,n}^d(t) && \forall d \in D \end{aligned} \tag{18}$$

and we reactualize the positions of the groups:

$$x_n^d(t+1) = x_n^d(t) + \Delta t\, v_n^d(t) \tag{19}$$

The third step consists in re-ordering the groups, i.e. in calculating the predecessor list at time $(t+1)\Delta t$: the predecessor group $(m,e) \overset{def}{=} \left(M(n,d,i;t+1), E(n,d,i;t+1)\right)$ of each group (n,d) on each lane *i*.

This step is relatively easy: although groups may overtake each other, the fraction of order permutations is low. For a group to overtake another group, it is necessary that the overtaken group has access to fewer lanes than the overtaking group, which allows the latter to use the extra lane.

For intersection we propose to adapt the methodology of Khoshyaran Lebacque 2008.

5.3. Special topics

The homogeneous case.

It can be shown that in the homogeneous case the previous scheme reduces to a lagrangian Godunov scheme for (9,10) (the difficult part is to show that the vehicle groups have the same velocity in both schemes).

It follows that in the homogeneous case the particle groups do not overtake each other, therefore step 3 is trivial, and the velocity of all particle groups is directly given by the equivalent fundamental diagram (10): $v_n^d(t) = V\left(\rho\left(x_n^d(t)\right)\right)$.

Thus step 1 becomes trivial too.

Thus the proposed scheme applies seamlessly to the homogeneous and the heterogeneous cases, and is well adapted to network modeling. The problem of intersection modeling will be treated in a sequel to this paper.

CFL (Courant-Friedrichs-Lewy) condition

This condition is given by

$$\Delta t \leq \min_{n,d\in D,i\in I^d}\left(\Delta N_n^d \big/ \rho_{max,i} W_{max,i}\right) \tag{20}$$

with $\rho_{max,i}, W_{max,i}$ being respectively the maximum density and the maximum backwards wave speed on lane $i \in I$. In the homogeneous case it can be shown that the maximum wave speed for the total density, W_{max}, satisfies:

$$W_{max} \leq \sum_{i\in I} \frac{\rho_{max}}{\rho_{max,i}} W_{max,i} \tag{21}$$

and therefore the CFL condition (20) simplifies to

$$\Delta t \leq \min_{n,d\in D,i\in I^d}\left(\Delta N_n^d\right) \big/ \rho_{max} W_{max} \tag{22}$$

6. Conclusion

The model presented in this paper is based on a macroscopic representation of traffic flow. The model does not take into account the dynamics of lane change, but aims at describing the result of lane changes. The resulting multi-lane model is based on a logit lane-assignment. It is multi-class, with a flux function defined implicitly, and it has been shown that it reduces to a standard LWR model in the homogeneous case, out of the range of influence of intersections and merges or diverges. The paper has proposed a lagrangian scheme for the numerical resolution of the model. Other lagrangian schemes could be proposed and will be described in a paper in preparation. Lagrangian schemes are particularly well-suited for the model, because they do not require the calculation of the gradient of the flux function.

Further work in preparation on this model includes the development of schemes for network modeling, and the completion of a method for the systematic validation of the model, particularly in the inhomogeneous case.

References

Buisson, C., Lebacque, J.P., Lesort, J.B. (1995-1996)
1. Macroscopic modelling of traffic flow and assignment in mixed networks. *Proc. of the Berlin ICCCBE Conf.* 1367-1374. (ed. PAHL, P.J., WERNER, H.). 1995.
2. STRADA, a discretized macroscopic model of vehicular flow in complex networks based on the Godunov scheme. *Proc. of the CESA'96 IEEE Conference*. 1996.
3. The STRADA model for dynamic assignment. *Proceedings of the 1996 ITS Conference*. 1996.

Daganzo C.F. (1997) A continuum theory of traffic dynamics for motorways with special lanes. *Transportation Research* B, 31, pp 83-102.

Greenberg J.M., Klar A., Rascle M. (2003) Congestion on Multilane highways. SIAM Journal of Applied Mathematics 63(3), 813-818.

Helbing D. (1997). Verkehrsdynamik. Springer Verlag.

Hoogendoorn, S.P., Bovy, P.H.L. (1999) Multiclass macroscopic traffic modelling : a multilane generalization using gas-kinetic theory. *Proceedings of the 14th ISTTT (International Symposium on Transportation and Traffic Flow Theory)*, A. Ceder Ed. pp 27-50. Jerusalem.

Jin, W.L. (2006). A kinematic wave theory of lane-changing vehicular traffic. *eprint arXiv:math/0503036*.

Khoshyaran M.M., Lebacque J.P. (2008) Lagrangian modelling of intersections for the GSOM generic macroscopic traffic flow model. *AATT, Athens 2008*.

Laval, J., Daganzo, (2005) C.F. Multi-lane hybrid traffic flow model : a theory on the impacts of lane-changing maneuvres. *TRB Annual*

Meeting 2005.

Laval, J., Daganzo, C.F. (2006) Lane-changing in traffic streams. *Transportation Research* 40B, 3, pp 251-264.

Laval J A and L Leclercq (2008). Microscopic modeling of the relaxation phenomenon using a macroscopic lane-changing model. *Transportation Research Part B*, 42 (6):511-522.

Lebacque, J.P., (1993), Les modèles macroscopiques de trafic. *Les Annales des Ponts* n°67, pp. 24-45

Lebacque, J.P., Khoshyaran, M.M. (2002) Macroscopic flow models. Presented at the *6th Meeting of the EURO Working Group on Transportation* 1998. Published in "*Transportation planning: the state of the art*" (Editors: M. Patriksson, M. Labbé), 119-139. Kluwer Academic Press.

Lebacque, J.P., Khoshyaran,M.M. (2009) A stochastic lane assignment scheme for macroscopic multi- lane traffic flow modelling. *TRISTAN VII*, Tromsø.

Lebacque J.P., Mammar S., Haj-Salem H. (2007) Generic second order traffic flow modeling. *Transportation and Traffic Flow Theory 2007* (R.E. Allsop, M.G.H. Bell, B.G. Heydecker Eds). London.

Lighthill, M.H., Whitham, G.B. On kinematic waves II : A theory of traffic flow on long crowded roads. *Proc. Royal Soc.* (Lond.) A 229 : 317-345. 1955.

Michalopoulos, P.G., Beskos, D.E., Yamauchi, Y. (1984) Multilane traffic flow models : some macroscopic considerations. *Transportation B*, 18, 377-395.

Ngoduy, D. (2006) *Macroscopic discontinuity modelling for multiclass multilane traffic flow operations*. TRAIL thesis series. Delft University Press.

Richards (1956), P.I. Shock-waves on the highway. *Opns. Res*. 4 : 42-51.

Schnetzler, B., Louis, X. Lebacque, J.P., (2010) *A multilane junction model*. Transportmetrica.

Schnetzler, B., (2011) private communication .